— Exercises —

1. Show that the equation $x_1^2 + x_1x_2 + x_2^2 = 27$ implicitly defines a function $x_2 = G(x_1)$ in a neighborhood of the point $(3, 3)$. Find G' expressed in terms of x_1 and x_2.

2. Textbook 9.4 p.434

3. **Regular points of a function.** Textbook 9.7 (b) (c) p.435

4. Consider the system in $\mathbb{R}^3$
$$\begin{cases} x^2 + y^2 - 2z^2 = 0 \\ x^2 + 2y^2 + z^2 = 4 \end{cases}$$
Show that for x close to 0, there exist positive functions $y(x)$ and $z(x)$ s.t. $(x, y(x), z(x))$ is a solution. Express y' in terms of x, y; express and z' in terms of x, z.

— Problems —

5. **A global implicit function.** Show that the equation $y^3+(x^2+1)y+x^4 = 0$ implicitly defines a function $\varphi : \mathbb{R} \to \mathbb{R}, x \mapsto y$ of class $\mathcal{C}^1$.

6. **The folium of Descartes.** Let c be a positive number. We defines $F(x, y) = x^3 + y^3 - 3cxy$ for every (x, y). The folium of Descartes is the curve $\mathcal{C}$ defined by $F(x, y) = 0$.
   (a) Find all points of $\mathcal{C}$ in a neighborhood of which you can solve the equation $F(x, y) = 0$ for y as a function $G(x)$ or for x as a function $H(y)$.
   (b) Find the derivative of G at the points found in the previous question. Find those values of x where $G'(x) = 0$. Same for H.
   (c) For every $t \neq -1$, we put $\phi(t) = (\frac{3ct}{1+t^3}, \frac{3ct^2}{1+t^3})$. Show that ϕ is a parametrization of the folium, i.e. that the curve $\mathcal{C}$ is the set $\phi(\mathbb{R} \setminus \{-1\})$.
   (d) Show that the line of equation $x + y = -c$ is an asymptote of $\mathcal{C}$. Sketch $\mathcal{C}$.